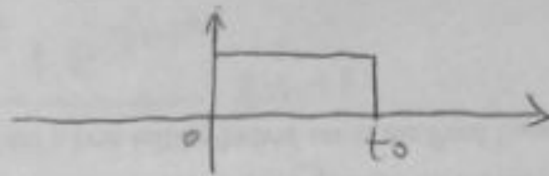
5.2

a) $\bar{F}(w) = \int_{-\infty}^{+\infty} k e^{-bt}[u(t) - u(t-t_0)] e^{-jwt} dt$

$u(t) - u(t-t_0)$
for $t \in (0, \infty)$



$$F(w) = \int_0^{t_0} k e^{-bt} e^{-jwt} dt$$

$$= \int_0^{t_0} k e^{-(jw+b)t} dt$$

$$= k \frac{[e^{-(jw+b)t}]\Big|_0^{t_0}}{-(jw+b)}$$

$$= k \frac{1 - e^{-(jwt_0 + bt_0)}}{b + jw}$$

b) $$F(w) = \int_{-\infty}^{+\infty} A \cos(w_0 t + \phi) e^{-jwt} dt$$

$$= A \int_{-\infty}^{\infty} \frac{e^{j(w_0 t + \phi)} + e^{-j(w_0 t + \phi)}}{2} e^{-jwt} dt$$

$$= \frac{A}{2} \int_{-\infty}^{+\infty} e^{j(w_0 t + \phi - wt)} + e^{-j(w_0 t + \phi + wt)} dt$$

$$= \frac{A}{2} \cancel{e} \int_{-\infty}^{+\infty} e^{j(w_0 - w)t + j\phi} + e^{-j(w_0 + w)t - j\phi} dt$$

$$= A\pi e^{j\phi} \delta(w - w_0) + A\pi e^{-j\phi} \delta(w + w_0)$$

time shift in frequency domain

Table 5.2
$A\delta(t - t_0) \rightarrow A e^{-jwt_0}$

5.6

a) $A e^{-\beta t} \cos(\omega_0 t)\, u(t)$

$$= A e^{-\beta t} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} u(t)$$

$$= \frac{A}{2}\left[e^{(j\omega_0 - \beta)t} + e^{-(j\omega_0 + \beta)t}\right] u(t)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$\exp(-\beta t) \to 0$ as $t \to \mathrm{Inf}$ because $\mathrm{Re}(\beta) > 0$

$$= \int_0^{\infty} \frac{A}{2}\left[e^{(j\omega_0 - \beta)t} + e^{-(j\omega_0 + \beta)t}\right] e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} \frac{A}{2}\left[e^{(j\omega_0 - j\omega - \beta)t} + e^{-(j\omega_0 + j\omega + \beta)t}\right] dt$$

$\frac{\mathrm{inf}}{A} + \frac{\mathrm{inf}}{-B}$ cancel out. $A/B = X + jY$

$$= \frac{A}{2} \frac{\left[e^{(j\omega_0 - j\omega - \beta)t}\right]\Big|_0^{\infty}}{j(\omega_0 - \omega) - \beta} + \frac{A}{2} \frac{\left[e^{-(j\omega_0 + j\omega + \beta)t}\right]\Big|_0^{\infty}}{-(j\omega_0 + j\omega + \beta)}$$

~~the~~ when $t \to \infty$. two inf cancels out.

$$= \frac{A}{2} \frac{-1}{j(\omega_0 - \omega) - \beta} + \frac{A}{2} \frac{-1}{-(j\omega_0 + \omega) - \beta}$$

$$= \frac{\frac{A}{2}}{\beta + j(\omega_0 + \omega)} + \frac{\frac{A}{2}}{\beta - j(\omega_0 - \omega)}$$

$$= \frac{\frac{A}{2}}{\beta + j(\omega + \omega_0)} + \frac{\frac{A}{2}}{\beta + j(\omega - \omega_0)}$$

$$F(\omega) = \underbrace{\frac{A/2}{\beta + j(\omega+\omega_0)}}_{p} + \underbrace{\frac{A/2}{\beta + j(\omega-\omega_0)}}_{q} \equiv p + q$$

let's consider,

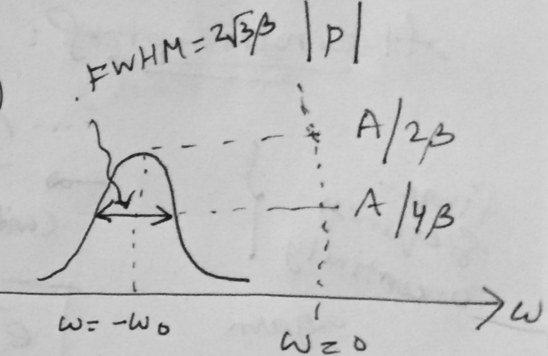
$$p = \frac{A/2}{\beta + j(\omega+\omega_0)}$$

$$\Rightarrow |p| = \frac{|A/2|}{|\beta + j(\omega+\omega_0)|}$$

$$\Rightarrow |p| = \frac{A/2}{\sqrt{\beta^2 + (\omega+\omega_0)^2}} \; : \text{Lorentzian}$$

when $\omega = -\omega_0$

$$|p| = \frac{A/2}{\sqrt{\beta^2 + 0^2}} = A/2\beta \leftarrow (\text{peak})$$



now lets consider $|p| = |p_{max}|/2$

$$\Rightarrow \frac{1}{2} \cdot A/2\beta = \frac{A/2}{\sqrt{\beta^2 + (\omega+\omega_0)^2}}$$

$$\Rightarrow \frac{1}{2\beta} = \frac{1}{\sqrt{\beta^2 + (\omega+\omega_0)^2}}$$

$$\Rightarrow 4\beta^2 = \beta^2 + (\omega+\omega_0)^2$$

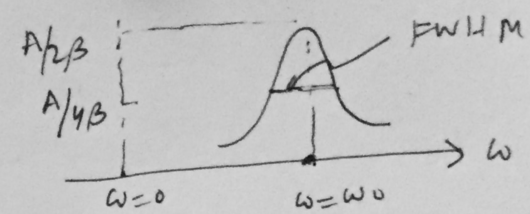
$$\Rightarrow (\omega+\omega_0)^2 = 3\beta^2$$

$$\Rightarrow (\omega+\omega_0) = \pm\sqrt{3}\beta$$

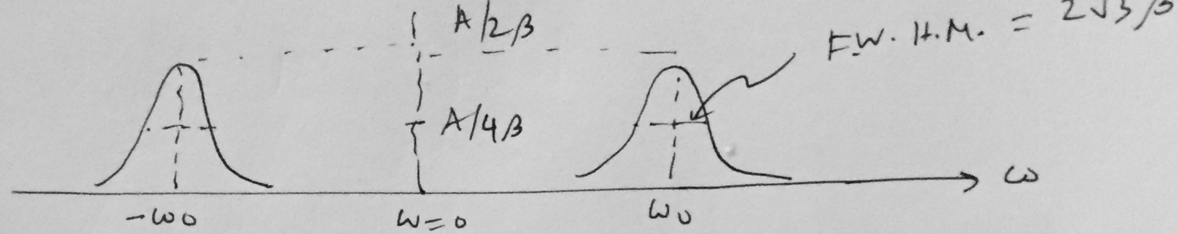
$$\Rightarrow \omega = -\omega_0 \pm \sqrt{3}\beta$$

$\Rightarrow$ Full width half maxima (FWHM) of this Lorentzian $= 2\sqrt{3}\beta$.

similarly, spectrum of $|q|$ is



and $|F(\omega)| = |p| + |q|$



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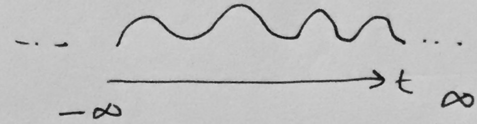
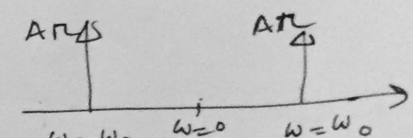
Alternative derivation: $\boxed{A\cos\omega_0 t \xrightarrow{F.T.} A\pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)}$

time-frequency uncertainity

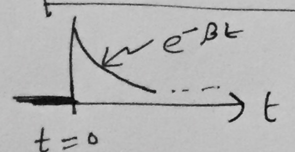


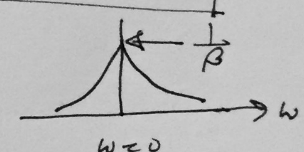
(wide spread in time)



(localized in frequency)

again $\boxed{e^{-\beta t}u(t) \xrightarrow{F.T.} \dfrac{1}{\beta+j\omega}}$





now, according to convolution theorem

$$f(t)\cdot g(t) \xrightarrow{F.T.} \frac{1}{2\pi} F(\omega) * G(\omega) \quad \left[\tfrac{1}{2\pi} \text{ is a scaling factor}\right]$$

$$\Rightarrow A e^{-\beta t}\cos\omega_0 t\, u(t) \xrightarrow{F.T.} \frac{1}{2\pi}\left\{ A\pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right) * \frac{1}{\beta+j\omega}\right\}$$

$$\Rightarrow \boxed{A e^{-\beta t}\cos\omega_0 t\, u(t) \xrightarrow{F.T.} \frac{A/2}{\beta+j(\omega-\omega_0)} + \frac{A/2}{\beta+j(\omega+\omega_0)}}$$

[sampling property of $\delta()$ function]

5.6 from Table 5.2

b)

$$A \sin(w_1 t) \xleftrightarrow{\mathcal{F}} \frac{A\pi}{j}\left[\delta(w-w_1) - \delta(w+w_1)\right]$$

$$B \cos(w_2 t) \xleftrightarrow{\mathcal{F}} B\pi\left[\delta(w-w_2) + \delta(w+w_2)\right]$$

$$\therefore F(w) = \frac{A\pi}{j}\left[\delta(w-w_1) - \delta(w+w_1)\right] + B\pi\left[\delta(w-w_2) + \delta(w+w_2)\right]$$

5.18 a) in next page

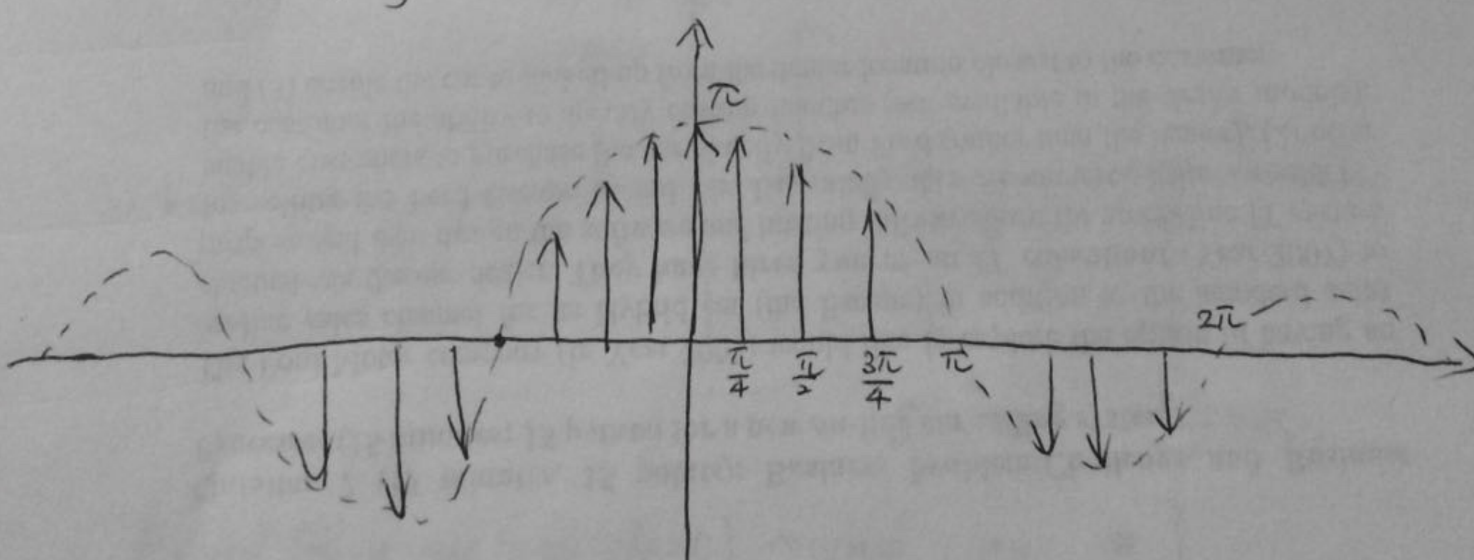
5.18

b) To (period) doubled $\quad T = 2 \quad T_0 = 8 \quad w_0 = \frac{\pi}{4}$.

$$F(w) = \sum_{n=-\infty}^{\infty} 2 \cdot 2 \cdot \frac{\pi}{4} \sin\left(n \cdot \frac{\pi}{4} \cdot \frac{2}{2}\right) \delta\left(w - n\frac{\pi}{4}\right)$$

$$= \sum_{n=-\infty}^{\infty} \pi \sin\left(\frac{n\pi}{4}\right) \delta\left(w - \frac{n\pi}{4}\right)$$

therefore the sinc function doesn't change while sampling rate doubles



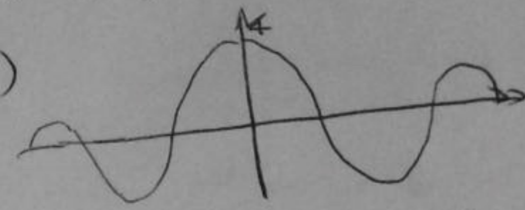
5.18 $A = 2 \quad T = 2 \quad T_0 = 4 \quad w_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$

a) for a single rect signal

$g(t) = A\,\mathrm{rect}(t/T) = 2\,\mathrm{rect}(t/2)$

$\Longleftrightarrow G(w) = AT\,\mathrm{sinc}(\frac{Tw}{2}) = 2 \cdot 2\,\mathrm{sinc}(w) = 4\,\mathrm{sinc}(w)$

(refer to book example 5.13)



for a periodical rect signal.

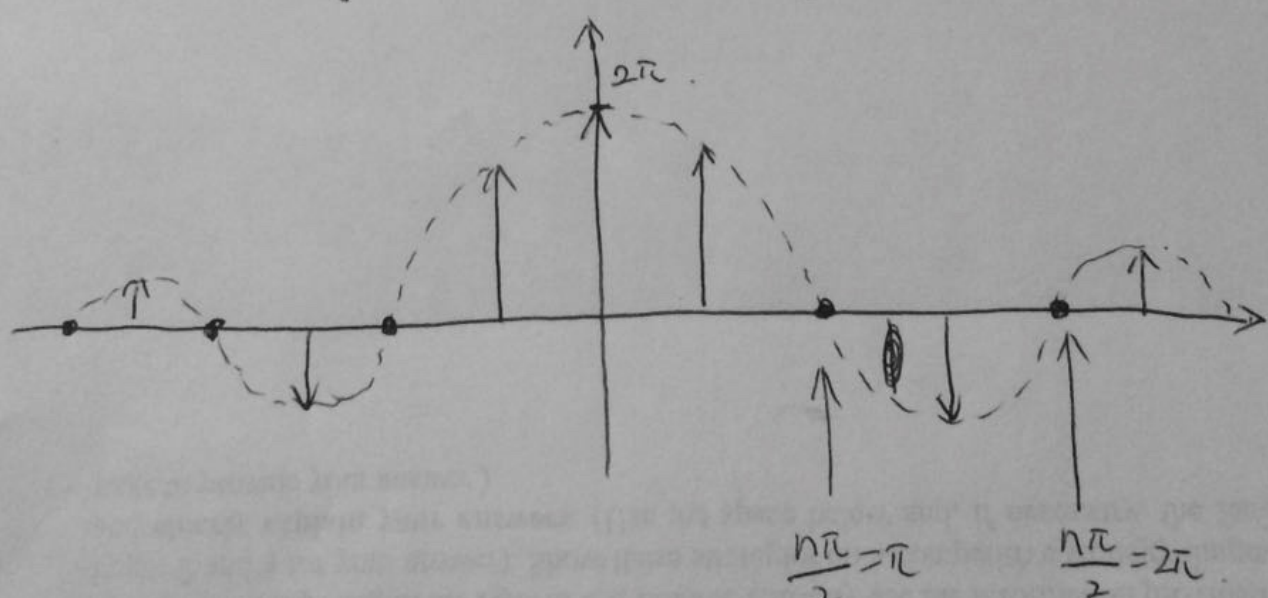
$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$ (note: n is integer)

$\Longleftrightarrow F(w) = \sum_{n=-\infty}^{\infty} w_0\,G(nw_0)\,\delta(w - w_0)$

$= \sum_{n=-\infty}^{\infty} \frac{\pi}{2} \cdot 4 \cdot \mathrm{sinc}(\frac{n\pi}{2})\,\delta(w - \frac{n\pi}{2})$

$= \sum_{n=-\infty}^{\infty} 2\pi \cdot \mathrm{sinc}(\frac{n\pi}{2})\,\delta(w - \frac{n\pi}{2})$.

A sinc graph with sampling points at x-axis $= \frac{n\pi}{2}$

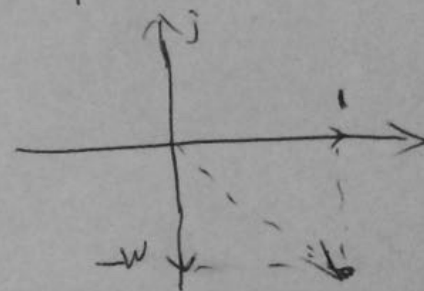


[signal @ x-axis (amplitude=0) marked as •]

5.12

i) $$\frac{V_2}{V_1} = \frac{\frac{1}{jwc}}{R + \frac{1}{jwc}} = \frac{1}{1 + jwRC} = \frac{1}{1 + jw}$$

ii) $$|H(w)| = \left|\frac{1}{1+jw}\right| = \left|\frac{(1-jw)}{(1+jw)(1-jw)}\right| = \left|\frac{1-jw}{1+w^2}\right|$$



$$\angle H(w) = \tan^{-1}\left(\frac{-w}{1}\right) = -\tan^{-1}(w)$$

$$|H(w)| = \frac{\sqrt{1^2 + w^2}}{1 + w^2} = \frac{1}{\sqrt{1 + w^2}}$$

iii) $$\because\ e^{-at}u(t) \longleftrightarrow \frac{1}{a + jw}$$

$$\therefore\ e^{-t}u(t) \longleftrightarrow \frac{1}{1 + jw}$$

$$\therefore\ h(t) = e^{-t}u(t)$$

$$H(j\omega) = \frac{1}{1+j\omega}$$